Topics in complex analysis   Lecture 10   (Nov 25, 2024)

<u>Goal</u> Define Riemann sphere $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$
(1-pt compactification, Alexandrov compactification)

<u>Def</u> (Topology) A <u>topology</u> $\mathcal{T} \subseteq 2^X$ on a given sp X is characterized by the following three properties

(i) $\emptyset \in \mathcal{T}$;

(ii) closedness under finite intersections: if $T_1, \dots, T_n \in \mathcal{T}$ then $T_1 \cap \dots \cap T_n \in \mathcal{T}$;

not nec countable!

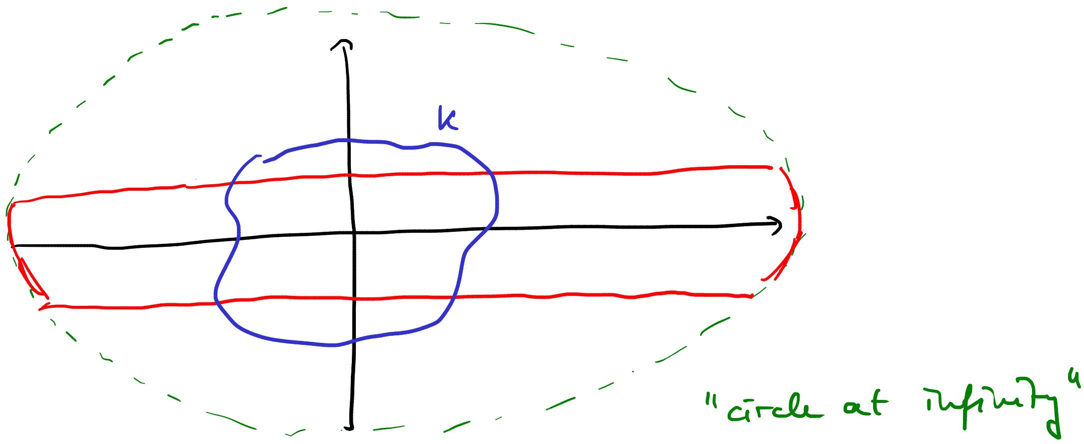
(iii) closedness under arbitrary unions: if T_i, $i \in I$, is in $\mathcal{T}$, so is $\bigcup_{i \in I} T_i$.

Let's adjoin an artificial point ∞ "at infinity" as follows.

<u>Def</u> (Riemann sphere) The Riemann sphere is the set $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ endowed with the top $\mathcal{T}$ defined as follows: a set $\hat{U} \subseteq \hat{\mathbb{C}}$ belongs to $\mathcal{T}$ iff

(i) $\hat{U}$ open in $\mathbb{C}$ if $\infty \notin \hat{U}$;

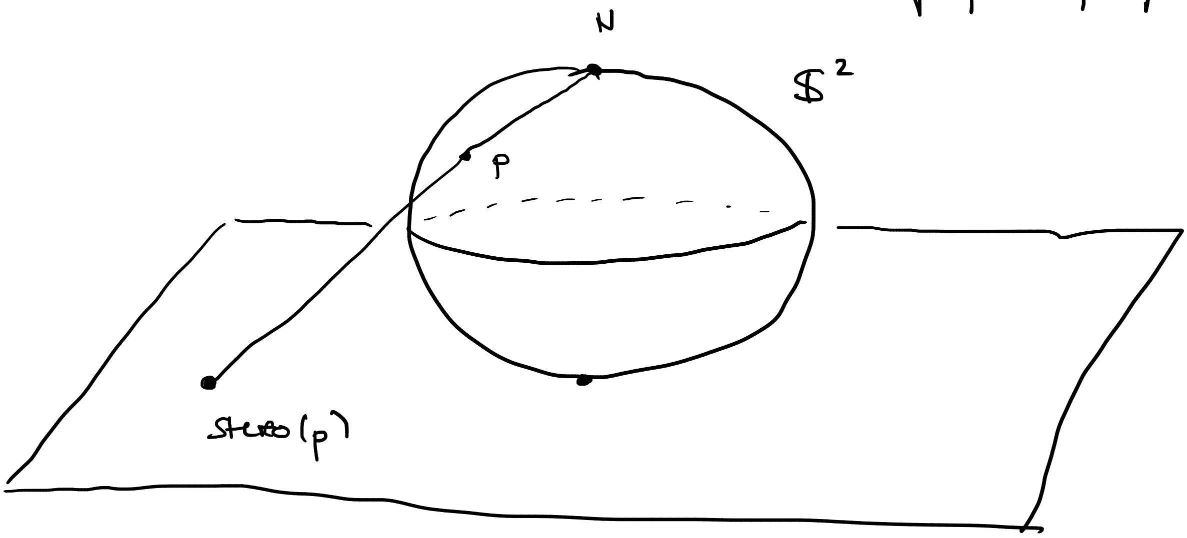
(ii) $\hat{U} \setminus \{\infty\} = \mathbb{C} \setminus K$ for some cpt $K \subseteq \underline{\mathbb{C}}$ if $\infty \in \hat{U}$.



Thm (Facts) • With this topology, $\hat{\mathbb{C}}$ is a compact metrizable sp. which is homeo to S^2

(induced by a metric)

"stereographic proj"



• A sequence $(z_n)_{n\in\mathbb{N}}$ in $\mathbb{C}$ converges to ∞ wrt this topology iff $1/z_n \longrightarrow 0$ as $n \longrightarrow \infty$

Theorem 7.3 (Identity theorem). *Let $\widehat{D} \subset \widehat{\mathbb{C}}$ be a domain and let $f, g : \widehat{D} \to \widehat{\mathbb{C}}$ be holomorphic. If the set $\{f = g\}$ has an accumulation point in $\widehat{D}$, then $f = g$.*

Proof Abbreviate $S := \{ z \in \hat{D} : f = g \text{ in a nbhd of } z \}$

Claim $S = \hat{D}$.

First S is open.

Next we show $S \neq \emptyset$. Let $z_0 \in \hat{D}$ be an accumulation pt of $\{f = g\}$ (by hypoth). Continuity implies $f(z_0) = g(z_0)$. We now apply the classical identity thm in the following cases ($r > 0$ suff small):

(i) $f, g : B_r(z_0) \longrightarrow \mathbb{C}$ if $z_0, f(z_0) \in \mathbb{C}$

(ii) $1/f, 1/g : B_r(z_0) \longrightarrow \mathbb{C}$ if $z_0 \in \mathbb{C}$, $f(z_0) = \infty$

(iii) $z \longmapsto f(1/z),\ g(1/z)$ on $B_r(z_0)$ if $z_0 = \infty$ and $f(z_0) \in \mathbb{C}$;

(iv) $z \longmapsto \dfrac{1}{f(1/z)},\ \dfrac{1}{g(1/z)}$ on $B_r(z_0)$ if $z_0 = f(z_0) = \infty$

From this we get $f = g$ in a nbhd of z_0.

Lastly, we show S is closed. Let $(s_n)_{n\in\mathbb{N}}$ be a sequence in S converging to some $s_0 \in \hat{D}$. If $s_0 \notin S$ then by def. of S this means s_0 is an acc pt of $\{f = g\}$. From the previous step, $f = g$ in a nbhd of s_0 $\Longrightarrow$ $s_0 \in S$ ↯ Hence $s_0 \in S$.

Connectedness of $\hat{D}$ yields $\hat{D} = S$. $\square$

Lemma 7.4. *Let $f : \widehat{\mathbb{C}} \to \mathbb{C}$ be holomorphic. Then f is constant.*

Proof: Since $\hat{\mathbb{C}}$ is cpt, so is $f(\hat{\mathbb{C}})$ (because every cts fct sends cpt sets to cpt sets). Since $f(\hat{\mathbb{C}}) \subseteq \mathbb{C}$ this means f is bdd. By Liouville thm, f is const. $\square$

Def (Möbius trafo) A Möbius (not always required) trafo is a fct $f : \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$ st $\exists\, a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$ st

$$f(z) = \begin{cases} \dfrac{az+b}{cz+d} & \text{if } z \in \mathbb{C}\setminus\{-d/c\} \\ \dfrac{a}{c} & \text{if } z = \infty \\ \infty & \text{if } z = -\dfrac{d}{c}. \end{cases}$$

Fact: The set of Möbius trafos with $\circ$ forms a group.

Corollary 7.8. *A function $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is biholomorphic if and only if f is a so-called Möbius transformation, i.e., there exist $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$ and*

$$f(z) = \frac{az + b}{cz + d}$$

with the convention that $f(\infty) = a/c \in \widehat{\mathbb{C}}$ and $f(-d/c) = \infty$.

Proof. Assume f is a Möbius trafo. Then by Rem 7.7, f is rational hence holomorphic on $\hat{\mathbb{C}}$. (It has its only pole at $-d/c$). Its inverse is given by

$$f^{-1}(z) = \begin{cases} \dfrac{dz - b}{-cz + a} & \text{if } z \in \mathbb{C} \setminus \{a/c\} \\ \infty & \text{if } z = \frac{a}{c} \\ -\frac{d}{c} & \text{if } z = \infty \end{cases}$$

This is a rational function, hence holo on $\hat{\mathbb{C}}$.
Hence f is biholomorphic.

Conversely let f be biholomorphic. If $f(\infty) = \infty$ then $f(\mathbb{C}) \subseteq \mathbb{C}$ by bijectivity and thus by Ex. f must be of the form $f(z) = az + b$, $a \neq 0$

Show $f : \mathbf{C} \to \mathbf{C}$ is holomorphic and injective if and only if f is of the form $f(z) = az + b$ for some $a \in \mathbf{C} \setminus \{0\}$ and $b \in \mathbf{C}$[3].

An affine map is clearly a Möbius trafo.

The case $f(\infty) \neq \infty$ is done by reduction to the previous case.

Consider the composition $\tilde{f} := \varphi \circ f$, where

$$\varphi(z) := \frac{1}{z - f(\infty)}$$

Thus $\tilde{f}$ is a biholomorphic fct with $\tilde{f}(\infty) = \infty$, hence it is affine by the previous step, hence $\tilde{f}(z) = az + b$ for some $a \neq 0$, $b \in \mathbb{C}$. Hence

$$f(z) = \varphi^{-1} \circ \tilde{f}(z) = \frac{-a}{-az - b}$$

is also a Möbius transformation.

(Alternatively: $\tilde{f}, \varphi^{-1}$ Möbius trafos $\Longrightarrow \varphi^{-1} \circ \tilde{f}$ Möbius trafo by the general group structure of the set of Möbius trafos.) $\square$